Dhofar University
Department of Electrical and Computer Engineering

Signals and Systems

Laplace Transform

Instructor
Dr Abdel-Rahman Al Qawasmi

Email: aalqawasmi@du.edu.om

"Principles of Signals and Systems by Orhan Gazi, 2023, Springer Nature,
ISBN: 978-3-031-17788-0.

Introduction:

In the preceding chapter, we have seen that the tools of Fourier analysis is extremely useful in the study of many problems of practical importance involving signals and LTI systems.
The generalization of the continuous-time Fourier transform is known as the Laplace transform.
As we will see, the Laplace transform has many of the properties that make Fourier analysis so useful.

5.1 The Laplace Transform:

The *Laplace transform* of a general signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$s = \sigma + j\omega$$

And the *inverse laplace transform* for $X(s)$ is given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

The above equations are referred as the Laplace transform pair.

$$x(t) \overset{L}{\Leftrightarrow} X(s)$$

If $s = j\omega$ ($\sigma = 0$) the Laplace transform of $x(t)$ becomes

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

which corresponds to the Fourier transform of $x(t)$.

Relationship between Fourier and Laplace Transforms:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = F\{x(t)e^{-\sigma t}\}$$

The Laplace transform of $x(t)$ is the Fourier transform of $x(t)e^{-\sigma t}$

Region of Convergence (ROC):

The Laplace transform of a general signal $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

In general, the range of values of s for which the integral in the above equation converges is referred to as the *region of convergence* (ROC) of the Laplace transform.

$X(s)$ can be converged for certain values of s even if the Fourier transform does not exist, because of the effect of the real exponential.

Example:

Find the Laplace transform for the exponential signal

$$x(t) = e^{-at}u(t)$$

Solution:

$$X(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(a+\sigma+j\omega)t}dt$$

$$= \frac{-1}{(a+\sigma)+j\omega} e^{-(a+\sigma+j\omega)t}\Big|_0^\infty$$

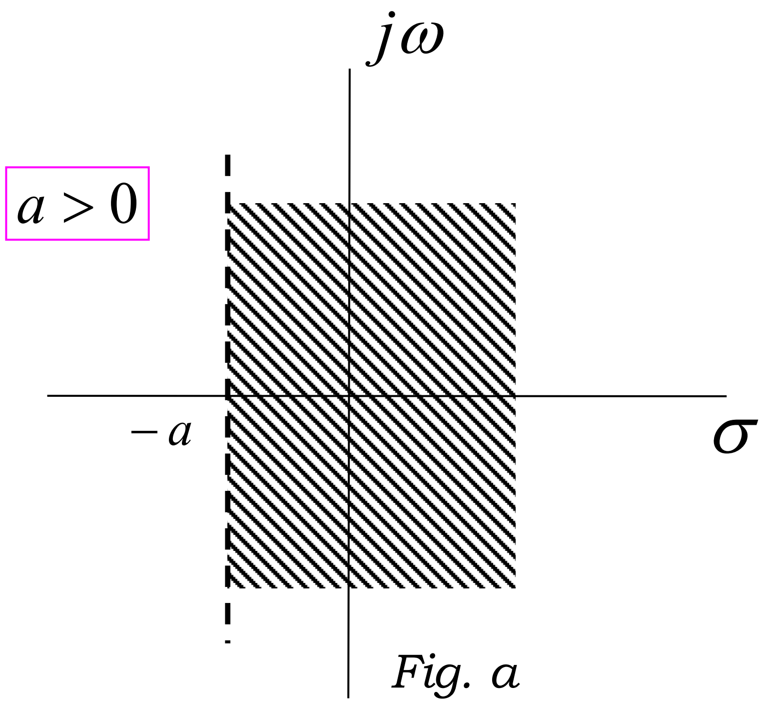
$$= \frac{1}{(a+\sigma+j\omega)} \quad , \qquad \text{if } \sigma > \text{-}a$$

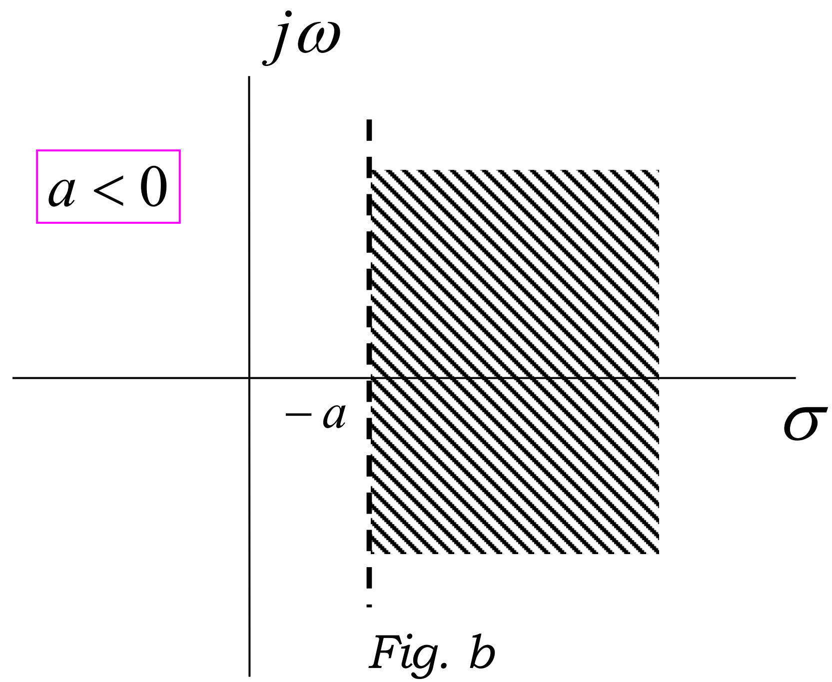
$$\sigma = \Re e\{s\}$$

$$= \frac{1}{a+s} \quad , \qquad \Re e\{s\} > \text{-}a$$

$$e^{-at}u(t) \overset{L}{\longleftrightarrow} \frac{1}{a+s} \quad , \quad \Re e\{\mathrm{s}\} > \text{-}a$$

For the above example, the ROC is shown in figures below:





We note that the F.T. for the signal in Fig.a exists because the $j\omega$ included in the ROC, but for the signal in Fig.b the F.T. does not exist

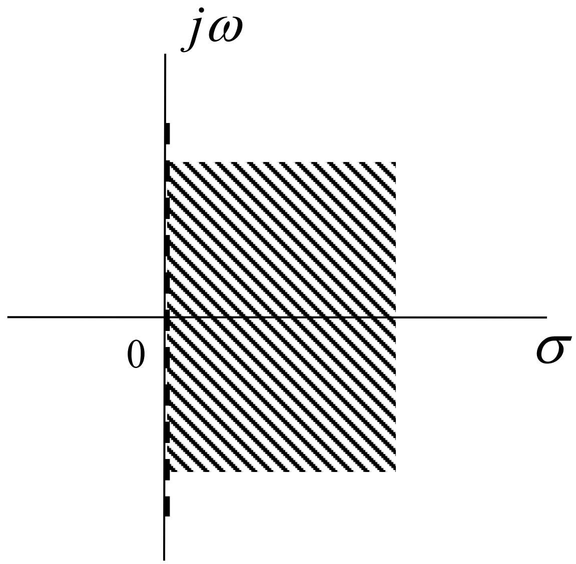
Example:

Find the Laplace transform for the unit step function

$$x(t) = u(t)$$

Solution:

$$u(t) \overset{L}{\longleftrightarrow} \frac{1}{s}, \quad \Re e\{s\} > 0$$



Example:

Find the Laplace transform for the exponential signal

$$x(t) = -e^{-at}u(-t)$$

Solution:

$$X(s) = \int_{-\infty}^{0} -e^{-at}e^{-st}dt = -\int_{-\infty}^{0} e^{-(a+\sigma+j\omega)t}dt$$

$$= \frac{1}{(a+\sigma)+j\omega} e^{-(a+\sigma+j\omega)t}\Big|_{-\infty}^{0}$$

$$= \frac{1}{(a+\sigma+j\omega)} \quad , \qquad \text{if } \sigma < \text{-}a$$

$$\sigma = \Re e\{s\}$$

$$= \frac{1}{a+s} \quad , \qquad \Re e\{s\} < \text{-}a$$

$$-e^{-at}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{a+s} \quad , \quad \Re e\{s\} < \text{-}a$$

For the above example, the ROC is shown in figures below:

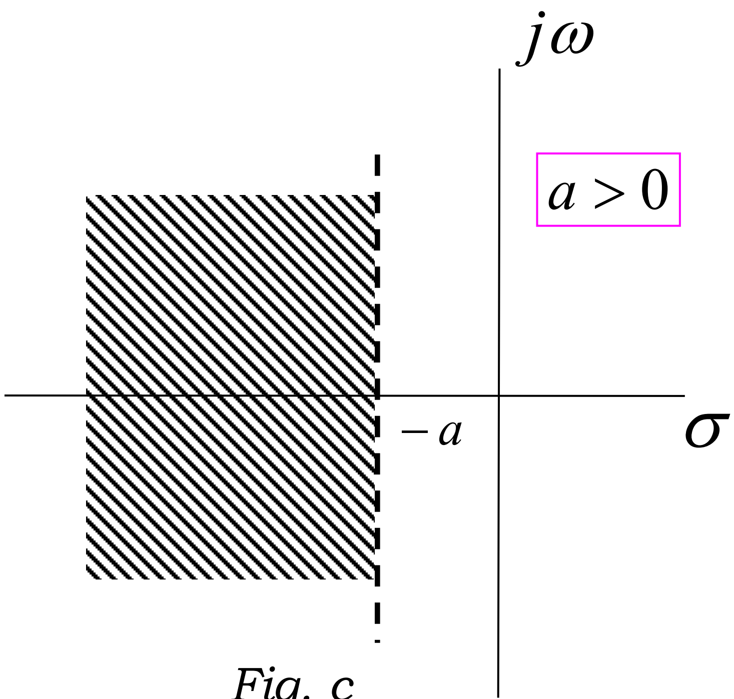


Fig. c

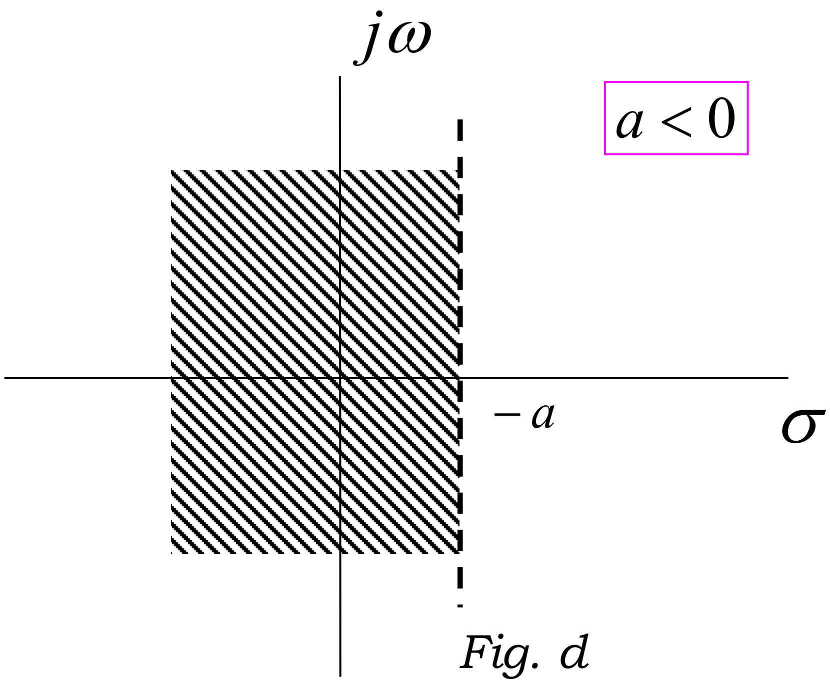


Fig. d

We note that the F.T. for the signal in Fig.d exists because the $j\omega$ included in the ROC, but for the signal in Fig.c the F.T. does not exist

Example:

Find the Laplace transform for the unit impulse function

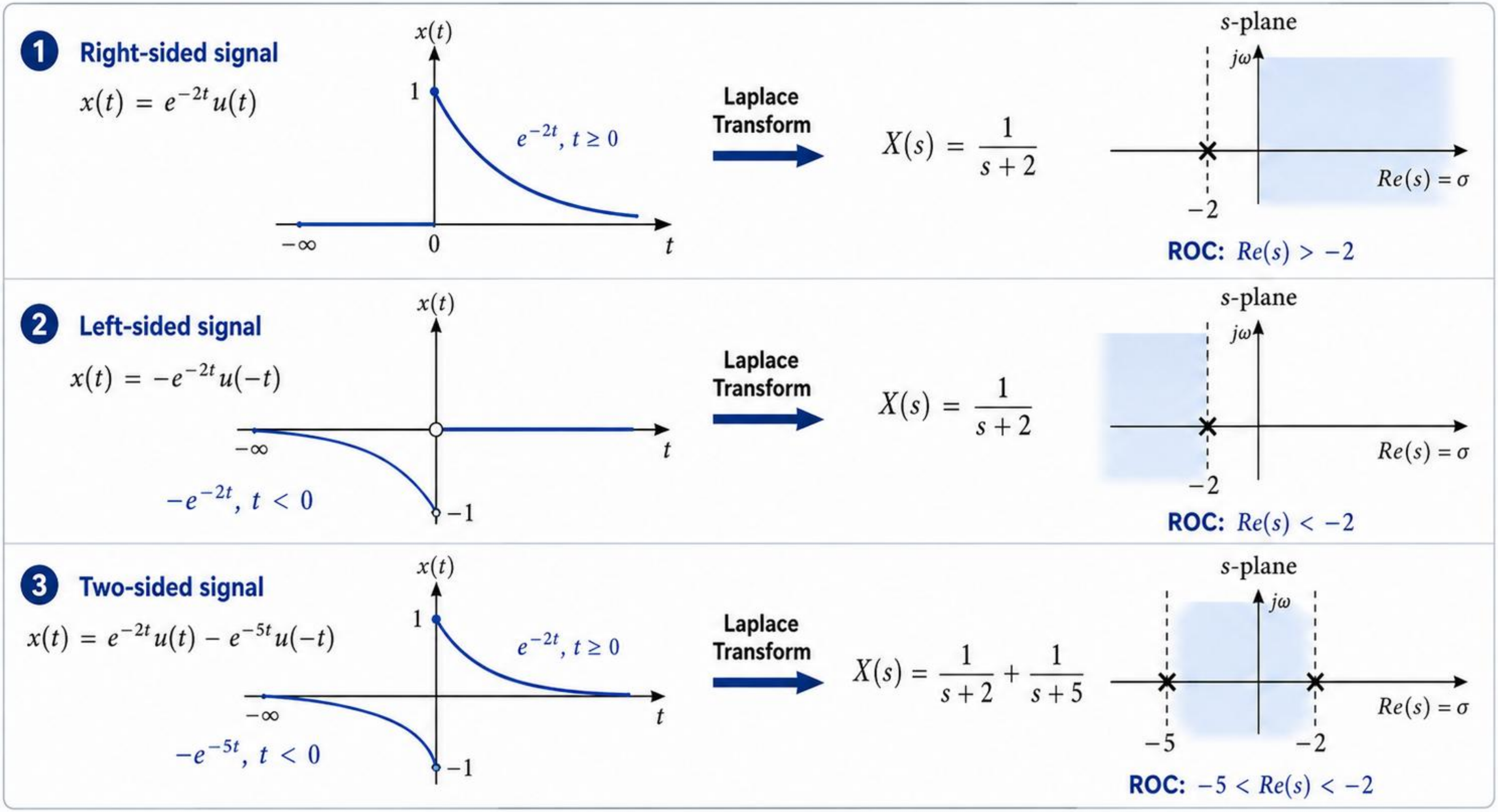
$$x(t) = \delta(t)$$

Solution:

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = e^{0} = 1 \quad , \quad \text{all } s$$

$$\boxed{\delta(t) \overset{L}{\longleftrightarrow} 1 \, , \quad all \; s}$$

Laplace Transform of Signals and the Region of Convergence (ROC)



Summary Rules

1. **Right-sided signal** → ROC to the right of the rightmost pole
2. **Left-sided signal** → ROC to the left of the leftmost pole
3. **Two-sided signal** → ROC between poles

Pole / Zero Diagram:

Let the Laplace transform of a signal $x(t)$ be a rational function in s, that is

$$X(s) = \frac{B(s)}{A(s)} \qquad \text{for } s \text{ in ROC}$$

where $B(s)$ and $A(s)$ are M-*th* and N-*th* order polynomials.

- The M roots of numerator $B(s)$ are called the ***zeros*** of the L.T.
- The N roots of denominator $A(s)$ are called the ***poles*** of the L.T.

- "x" is used to indicate poles and "o" is used to indicate zeros.

- The rational $B(s)/A(s)$ is unbounded for the poles of the L.T. Therefore, the poles of $B(s)/A(s)$ lie outside the ROC, the zeros may lie inside or outside the ROC.

Example:

Find the Laplace transform of the signal

$$x(t) = e^{-at}u(t) + e^{-bt}u(t) \quad , \quad a \neq b$$

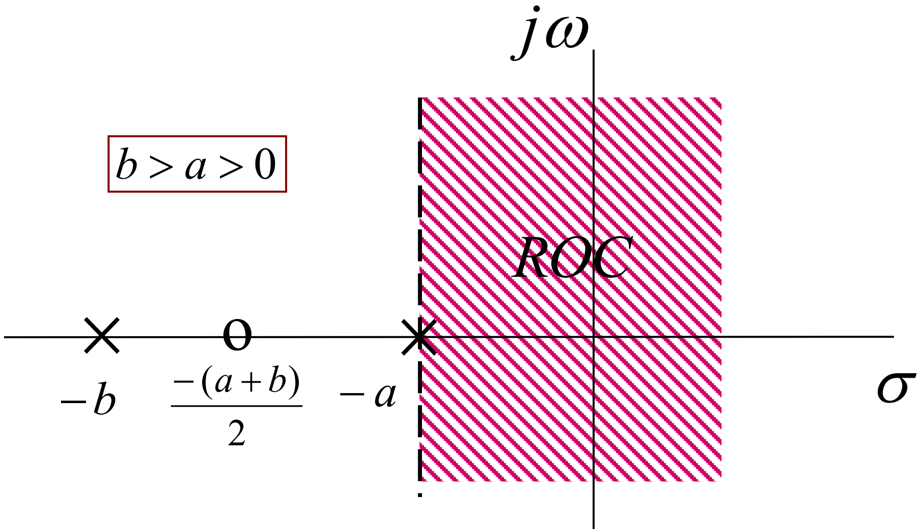
Solution:

$$X(s) = \frac{1}{s+a} + \frac{1}{s+b} = \frac{2s+a+b}{(s+a)(s+b)} \quad , \quad \text{Re}\{s\} > \max(-a, -b)$$

The poles at $s = -a$ and $s = -b$

The zero at $s = -(a+b)/2$

There is additional zero when $s \to \infty$



Example:

Find the Laplace transform for the causal sinusoid

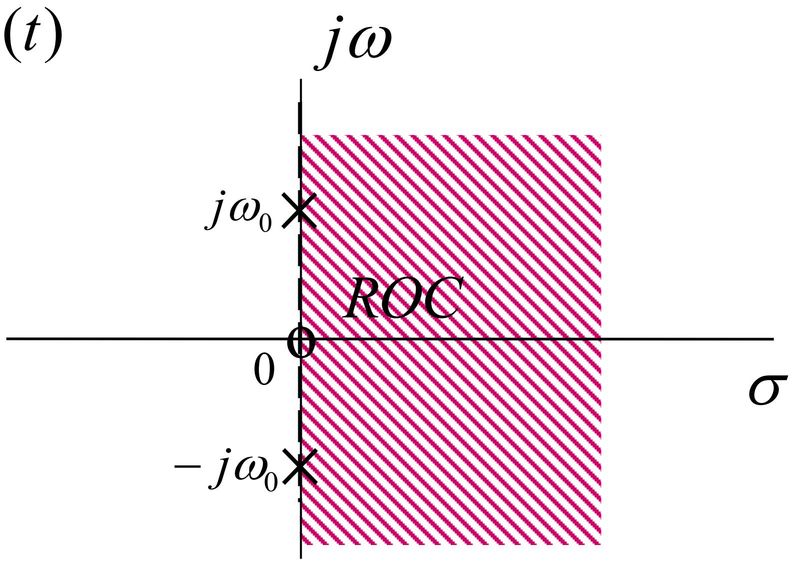
$$x(t) = [\cos \omega_0 t] u(t)$$

Solution:

$$x(t) = [\cos \omega_0 t] u(t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] u(t)$$

$$X(s) = \frac{1}{2}\frac{1}{s - j\omega_0} + \frac{1}{2}\frac{1}{s + j\omega_0}$$

$$= \frac{s}{(s^2 + \omega_0^2)}, \quad \text{Re}\{s\} > 0$$



Example:

Find the Laplace transform for the causal sinusoid

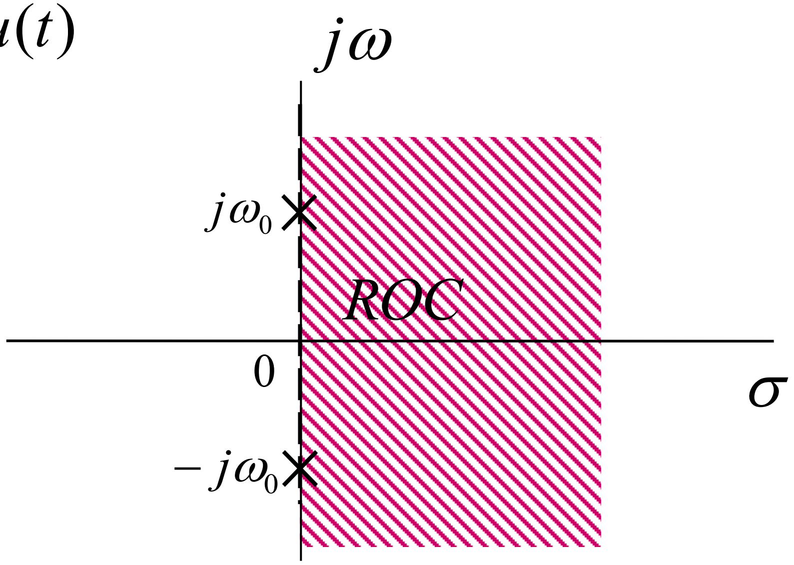
$$x(t) = [\sin \omega_0 t]u(t)$$

Solution:

$$x(t) = [\sin \omega_0 t]u(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]u(t)$$

$$X(s) = \frac{1}{2j}\frac{1}{s - j\omega_0} - \frac{1}{2j}\frac{1}{s + j\omega_0}$$

$$= \frac{\omega_0}{(s^2 + \omega_0^2)}\ , \qquad \mathrm{Re}\{s\} > 0$$



Laplace Transform Table

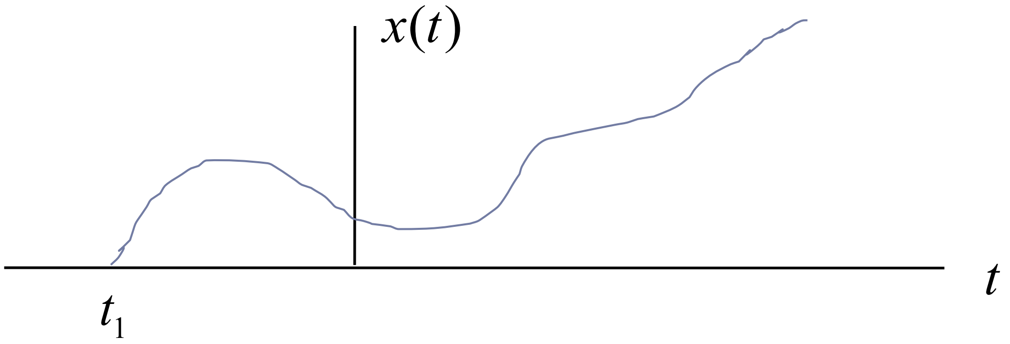
Signal	$x(t)$	$X(s)$	ROC
Exponential	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > -a$
	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < -a$
Impulse	$\delta(t)$	1	All s
Unit step	$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
	$-u(-t)$	$\frac{1}{s}$	$\text{Re}\{s\} < 0$
Cosine wave	$[\cos(\omega_0 t)]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
Sine wave	$[\sin(\omega_0 t)]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
Weighted exponential	$te^{-at}u(t),\ a>0$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > -a$
	$e^{-a\|t\|}$	$\frac{2a}{a^2-s^2}$	$-a < \text{Re}\{s\} < a$

5.2 ROC Properties:

- If $x(t)$ is a right sided signal [$x(t)=0,\ t < t_1$] and $X(s)$ is converges for some value of s, then the ROC must be of the form

$$\text{ROC}: \quad \Re e\{s\} > \sigma_{\max}$$

Where $\sigma_{\max}$ is the maximum real part of any of the poles.



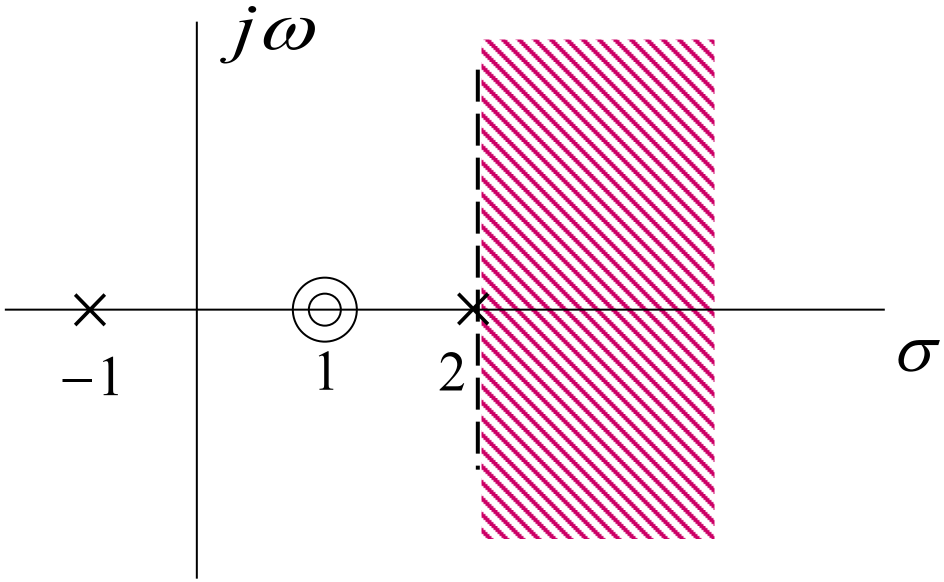
Example: (Right-sided signal)

Find the Laplace transform for the signal

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

Solution:

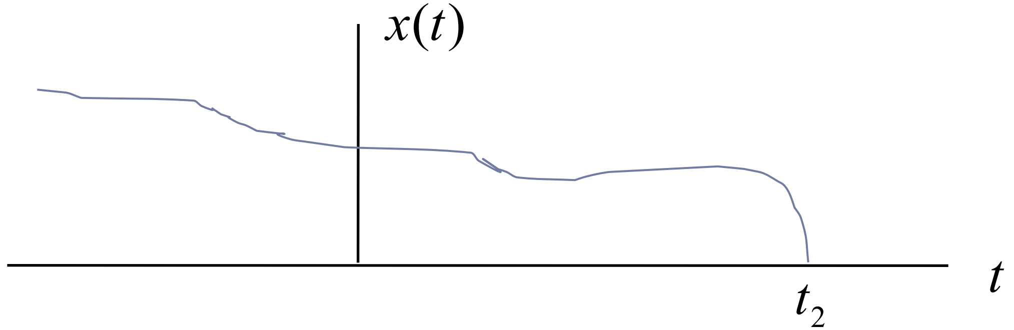
$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)} \quad , \quad \mathrm{Re}\{s\} > 2$$



- If $x(t)$ is a left sided signal [$x(t)=0,\ t > t_2$] and $X(s)$ is converges for some value of s, then the ROC must be of the form

$$\text{ROC}: \qquad \Re e\{s\} < \sigma_{\min}$$

Where $\sigma_{\min}$ is the minimum real part of any of the poles.



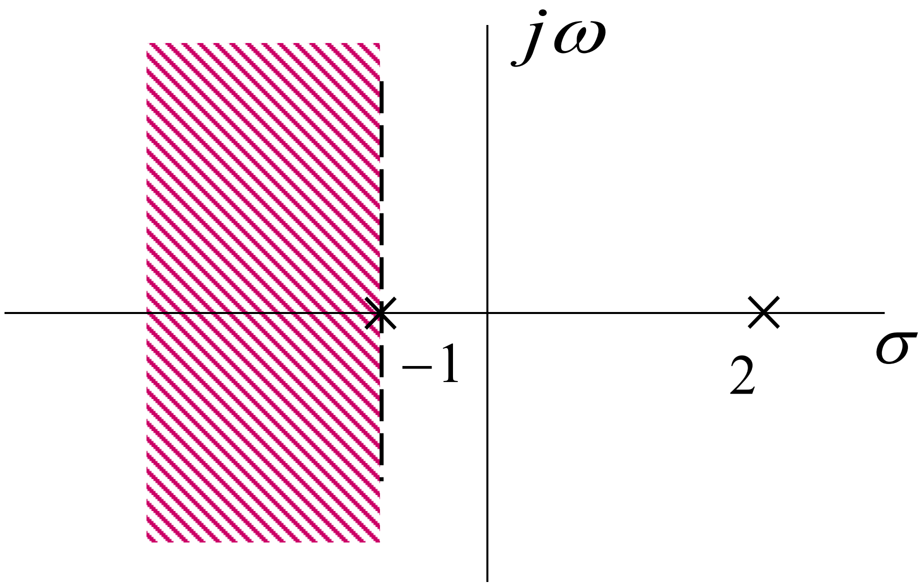
Example: (Left-sided signal)

Find the Laplace transform for the signal

$$x(t) = 2e^{-t}u(-t) - 3e^{2t}u(-t)$$

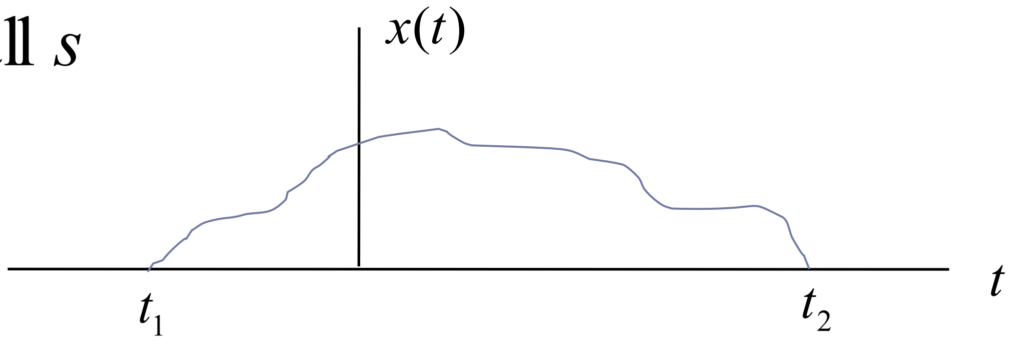
Solution:

$$X(s) = -2\frac{1}{s+1} + 3\frac{1}{s-2} \quad , \qquad \mathrm{Re}\{s\} < -1$$



- If $x(t)$ is a finite duration signal and $X(s)$ is converges for some value of s, then the ROC must be of the form

$$\text{ROC}: \quad \text{all } s$$



Example:

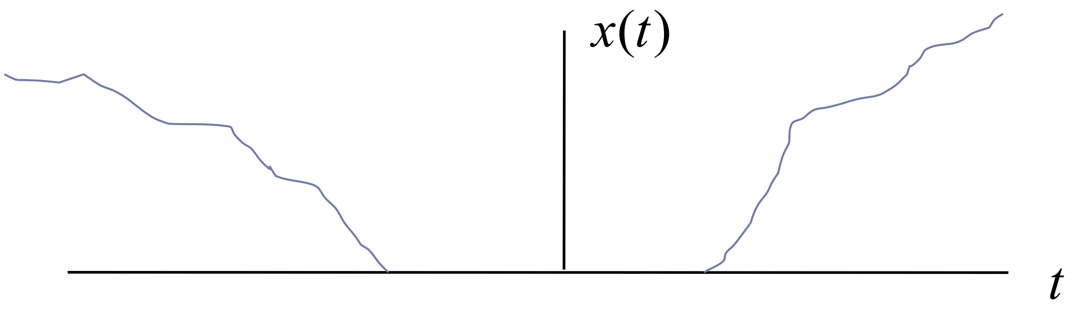
$$x(t) = \delta(t)$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = e^{0} = 1 \quad , \quad \text{all } s$$

- If *x(t)* is a two sided signal and *X(s)* is converges for some value of *s*, then the ROC must be of the form

$$\text{ROC}: \qquad \sigma_1 < \Re e\{s\} < \sigma_2$$

Where σ_1 and σ_2 are the real parts of two of the poles.



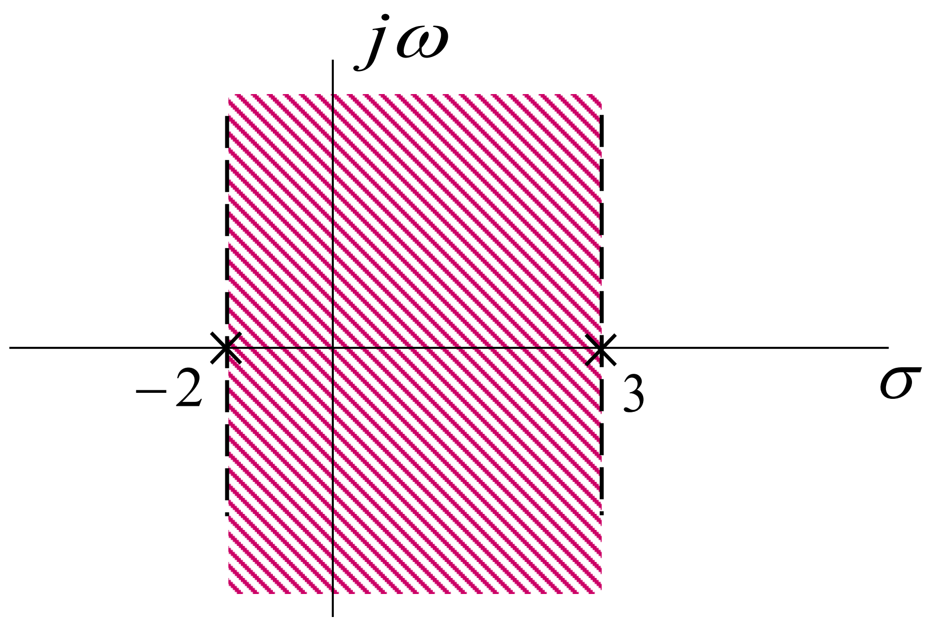
Example: (Two-sided signal)

Find the Laplace transform for the signal

$$x(t) = e^{-2t}u(t) - e^{3t}u(-t)$$

Solution:

$$X(s) = \frac{1}{s+2} + \frac{1}{s-3} \quad , \qquad -2 < \mathrm{Re}\{s\} < 3$$



Example: (Two-sided signal)

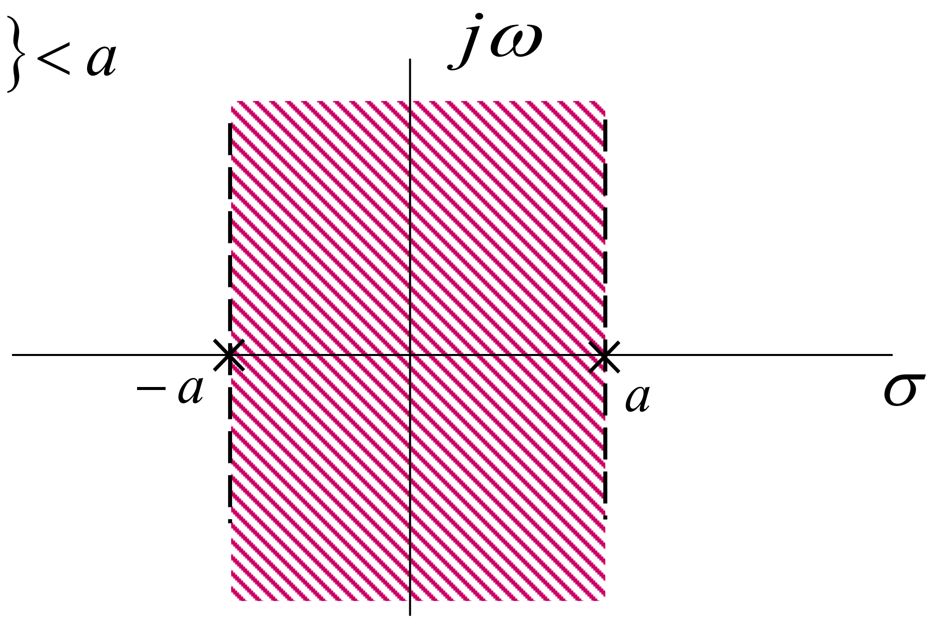
$$x(t) = e^{-a|t|} \quad , \qquad a > 0$$

Solution:

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$X(s) = \frac{1}{s+a} - \frac{1}{s-a} \quad , \qquad -a < \mathrm{Re}\{s\} < a$$

$$= \frac{2a}{a^2 - s^2} \quad , \qquad -a < \mathrm{Re}\{s\} < a$$



5.3 The Inverse Laplace Transform:

The *inverse laplace transform* for $X(s)$ is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

The above integral can be used to find the inverse Laplace transform using the theory of complex variables. However, for a Laplace transform, whose algebraic form is a rational fraction, a much simpler technique based upon partial fraction expansion can be used.

Example: Find $x(t)$ for

$$X(s) = \frac{2}{s^2 + 4s + 3}$$

Solution: (Partial-fraction expansion)

$$\frac{2}{s^2 + 4s + 3} = \frac{2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3) + B(s+1)}{(s+1)(s+3)}$$

To find A and B: $\quad 2 = A(s+3) + B(s+1)$

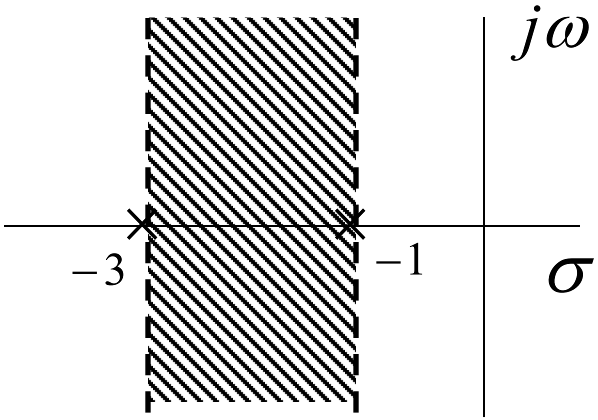
$$\text{let} \quad s = -1 \Rightarrow 2 = A(-1+3) \Rightarrow \quad A = 1$$

$$\text{let} \quad s = -3 \Rightarrow 2 = B(-3+1) \Rightarrow \quad B = -1$$

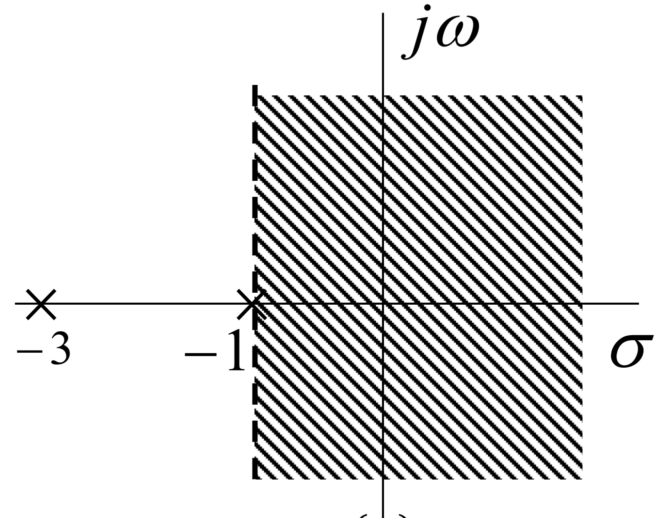
$$X(s) = \frac{1}{s+1} - \frac{1}{s+3} \,, \qquad \text{poles at } s = \text{-1} \quad \text{and} \quad s = \text{-3}$$

Solution: (continued)

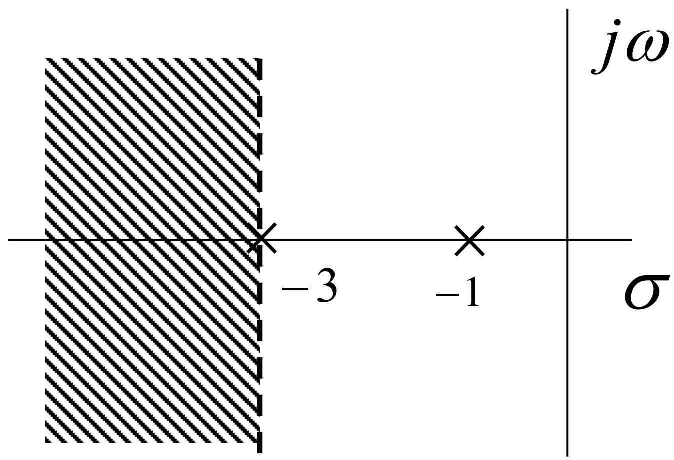
There are three possible forms of ROC



$$-3 < \Re e\{s\} < -1$$



$$\Re e\{s\} > -1$$



$$\Re e\{s\} < -3$$

Solution: (continued)

Case(1): If the ROC is $\Re e\{s\} > -1$ then *x(t)* must be right sided

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1} \ , \qquad \Re e\{s\} > -1$$

and

$$e^{-3t}u(t) \leftrightarrow \frac{1}{s+3} \ , \qquad \Re e\{s\} > -3$$

then

$$X(s) = \frac{1}{s+1} - \frac{1}{s+3} \ , \qquad \Re e\{s\} > -1$$

and

$$x(t) = \left[e^{-t} - e^{-3t}\right]u(t)$$

Solution: (continued)

Case(2): If the ROC is $\Re e\{s\} < -3$ then *x(t)* must be left sided

$$-e^{-t}u(-t) \leftrightarrow \frac{1}{s+1}\ , \qquad \Re e\{s\} < -1$$

and

$$-e^{-3t}u(-t) \leftrightarrow \frac{1}{s+3}\ , \qquad \Re e\{s\} < -3$$

then

$$X(s) = \frac{1}{s+1} - \frac{1}{s+3}\ , \qquad \Re e\{s\} < -3$$

and

$$x(t) = \left[-e^{-t} + e^{-3t}\right]u(-t)$$

Solution: (continued)

Case(3): If the ROC is $-3 < \Re e\{s\} < -1$ then $x(t)$ must be two sided

$$-e^{-t}u(-t) \leftrightarrow \frac{1}{s+1}\ , \qquad \Re e\{s\} < -1$$

and

$$e^{-3t}u(t) \leftrightarrow \frac{1}{s+3}\ , \qquad \Re e\{s\} > -3$$

then

$$X(s) = \frac{1}{s+1} - \frac{1}{s+3}\ , \qquad -3 < \Re e\{s\} < -1$$

and

$$x(t) = -e^{-t}u(-t) - e^{-3t}u(t)$$

Example: Find $x(t)$ for

Solution:

$$X(s)=\frac{s^2-2}{s^2+2s}$$

By division $X(s)=1+\dfrac{-2s-2}{s^2+2s}$

$$\begin{array}{r|l} & 1 \\ s^2+2s & s^2-2 \\ & \underline{s^2+2s} \\ & -2s-2 \end{array}$$

Partial-fraction expansion

$$\frac{-2s-2}{s^2+2s}=\frac{A}{s}+\frac{B}{s+2}=\frac{A(s+2)+Bs}{s(s+2)}$$

Find A and B: $-2s-2=A(s+2)+Bs$

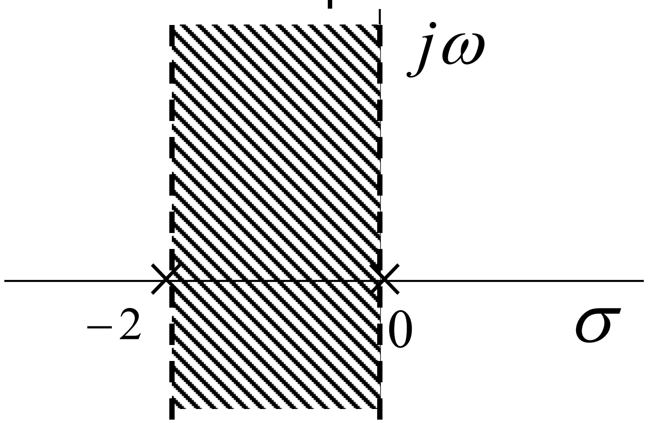
let $s=0 \Rightarrow -2=A(0+2)+B(0) \Rightarrow A=-1$

let $s=-2 \Rightarrow -2(-2)=A(-2+2)+B(-2) \Rightarrow B=-1$

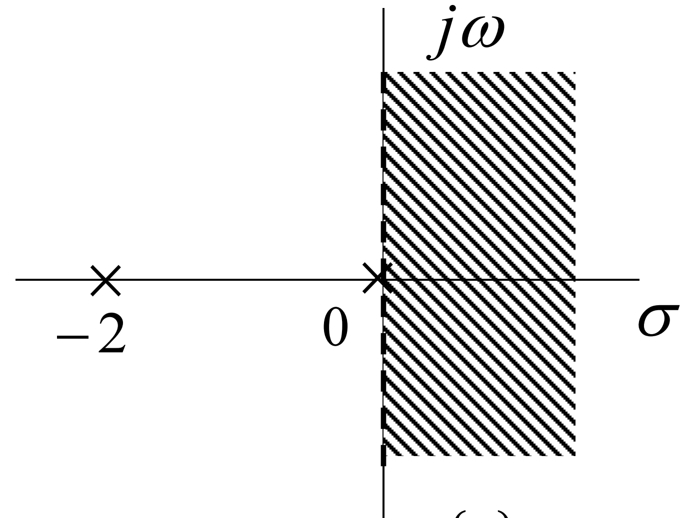
$X(s)=1-\dfrac{1}{s}-\dfrac{1}{s+2}$, poles at $s=0$ and $s=-2$

Solution: (continued)

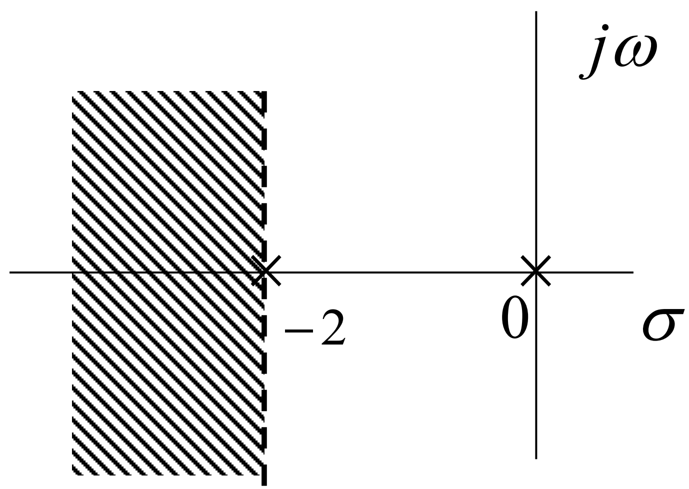
There are three possible forms of ROC



$$-2 < \Re e\{s\} < 0$$



$$\Re e\{s\} > 0$$



$$\Re e\{s\} < -2$$

Solution: (continued)

$$X(s) = 1 - \frac{1}{s} - \frac{1}{s+2} \;, \qquad \text{poles at } s = 0 \text{ and } s = -2$$

Case(1): If the ROC is $\Re e\{s\} > 0$ then *x(t)* must be right sided

$$x(t) = \delta(t) - u(t) - e^{-2t}u(t)$$

Case(2): If the ROC is $\Re e\{s\} < -2$ then *x(t)* must be left sided

$$x(t) = \delta(t) + u(-t) + e^{-2t}u(-t)$$

Case(3): If the ROC is $-2 < \Re e\{s\} < 0$ then *x(t)* must be two sided

$$x(t) = \delta(t) + u(-t) - e^{-2t}u(t)$$

5.4 Properties of the Laplace Transform:

The Laplace transform properties closely parallel for the Fourier Transform, as might be expected, since the Laplace transform can be viewed as the Fourier of the signal $x(t)e^{-\sigma t}$

The Laplace transform pair can be denoted as

$$x(t) \leftrightarrow X(s)$$

or

$$X(s) = L\{x(t)\}$$

where $L\{\cdot\}$ denotes the Laplace transform operation

1. Linearity:

If	$x_1(t) \leftrightarrow X_1(s)$	with ROC R_1
and	$x_2(t) \leftrightarrow X_2(s)$	with ROC R_2
then	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$	with ROC containing $R_1 \cap R_2$

Example:

$$4e^{-3t}u(t) - 2e^{-6t}u(t) \xleftrightarrow{L} \frac{4}{s+3} - \frac{2}{s+6} \quad , \quad \Re e\{s\} > -3$$

2. Time Shifting:

$$\text{If} \quad x(t) \stackrel{L}{\longleftrightarrow} X(s) \qquad \text{with ROC} = R$$

$$\text{then} \quad x(t-t_o) \stackrel{L}{\longleftrightarrow} e^{-st_o}X(s) \qquad \text{with ROC} = R$$

Example:

$$e^{-3(t-2)}u(t-2) \stackrel{L}{\longleftrightarrow} \frac{1}{s+3}e^{-2s} \qquad \Re e\{s\} > -3$$

3. Modulation (Frequency Shifting):

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with $\text{ROC} = R$

then $x(t)e^{s_0 t} \overset{L}{\longleftrightarrow} X(s - s_0)$ with $\text{ROC} = R + \Re e\{s_0\}$

Examples:

$a-$ $x(t)e^{j\omega_0 t} \overset{L}{\longleftrightarrow} X(s - j\omega_0)$ with $\text{ROC} = R$

$b-$ $e^{-4t}u(t)e^{3t} \overset{L}{\longleftrightarrow} \dfrac{1}{4 + (s-3)}$ $\Re e\{s\} > -1$

4. Time / Frequency Scaling:

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with ROC $= R$

then $x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$ with ROC $R_1 = aR$

Example:

$$x(-2t) \overset{L}{\longleftrightarrow} \frac{1}{2} X\left(\frac{-s}{2}\right) \quad \text{with ROC} = -2R$$

Consequence:

$$x(-t) \overset{L}{\longleftrightarrow} X(-s) \quad \text{with ROC} = -R$$

Example:

$$e^{4t}u(-t) \overset{L}{\longleftrightarrow} \frac{1}{4-s} \quad \Re e\{s\} < 4$$

5. Differentiation and Integration:

If $\quad x(t) \overset{L}{\longleftrightarrow} X(s) \quad$ with ROC $= R$

then $\quad \dfrac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s) \quad$ with ROC containing R

and $\quad \displaystyle\int_{-\infty}^{t} x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{1}{s}X(s) \quad$ with ROC containing $R \cap \Re e\{s\} > 0$

Example:

$$a- \qquad \frac{d}{dt}[e^{-4t}u(t)] \overset{L}{\longleftrightarrow} s\frac{1}{4+s} \qquad \Re e\{s\} > -4$$

$$b- \qquad \int_{-\infty}^{t} e^{-4t}u(t)dt \overset{L}{\longleftrightarrow} \frac{1}{s(4+s)} \qquad \Re e\{s\} > 0$$

6. Convolution of Signals:

If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with $\text{ROC} = R_1$

and $h(t) \overset{L}{\longleftrightarrow} H(s)$ with $\text{ROC} = R_2$

then $x(t) * h(t) \overset{L}{\longleftrightarrow} X(s) \cdot H(s)$ with ROC containing $R_1 \cap R_2$

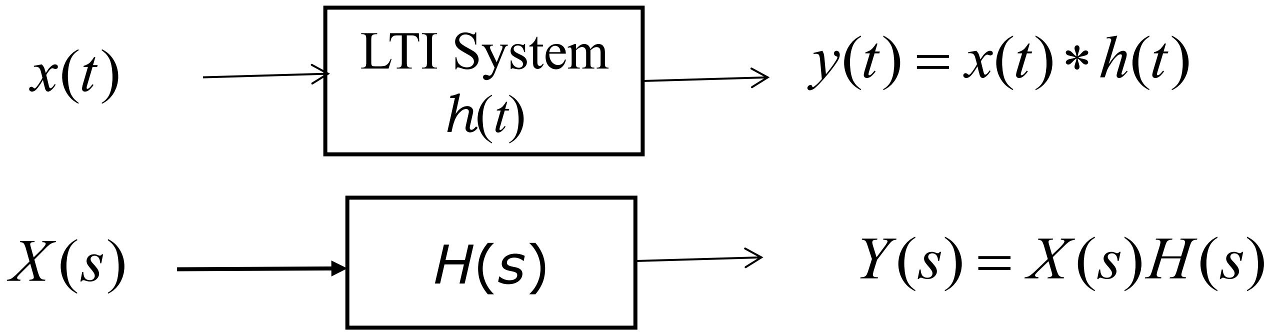
Example:

Let $X(s) = \dfrac{s+1}{s+3}$ $\quad \Re e\{s\} > -3$

and $H(s) = \dfrac{s+4}{s+1}$ $\quad \Re e\{s\} > -1$

then $X(s)H(s) = \dfrac{s+4}{s+3}$ $\quad \Re e\{s\} > -3$

5.5 The System Function of LTI Systems:



- The output of the LTI system is given by:

$$y(t) = x(t) * h(t)$$

- according to convolution property (F.T. properties)

$$Y(s) = X(s)H(s)$$

- where $H(s)$ is called the system function or (transfer function) of the system

$$H(s) = \frac{Y(s)}{X(s)}$$

Example: Find the output of the LTI system with

$$h(t) = e^{-2t}u(t) \quad \text{and} \quad x(t) = e^{-3t}u(t)$$

Solution:

$$X(s) = \frac{1}{3+s} \quad \Re e\{s\} > -3$$

$$H(s) = \frac{1}{2+s} \quad \Re e\{s\} > -2$$

$$\Rightarrow \quad Y(s) = \frac{1}{(2+s)(3+s)} \quad \Re e\{s\} > -2$$

partial fraction :

$$\frac{1}{(2+s)(3+s)} = \frac{A}{(2+s)} + \frac{B}{(3+s)}$$

then $A = 1$ and $B = -1$

using partial fraction

$$Y(s) = \frac{1}{2+s} - \frac{1}{3+s} \quad \Rightarrow \quad y(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

Example: Find the step response for the LTI system with

$$h(t) = e^{-2t}u(t)$$

Solution:

$$x(t) = u(t) \quad \Rightarrow \quad X(s) = \frac{1}{s} \quad \Re e\{s\} > 0$$

$$H(s) = \frac{1}{2+s} \quad \Re e\{s\} > -2$$

$$\Rightarrow \quad S(s) = \frac{1}{s(2+s)} \quad \Re e\{s\} > 0$$

using partial fraction

$$S(s) = \frac{-1}{2(2+s)} + \frac{1}{2s} \qquad \Re e\{s\} > 0$$

$$\Rightarrow s(t) = \frac{-1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) = \frac{1}{2}\left[1 - e^{-2t}\right]u(t)$$

partial fraction :

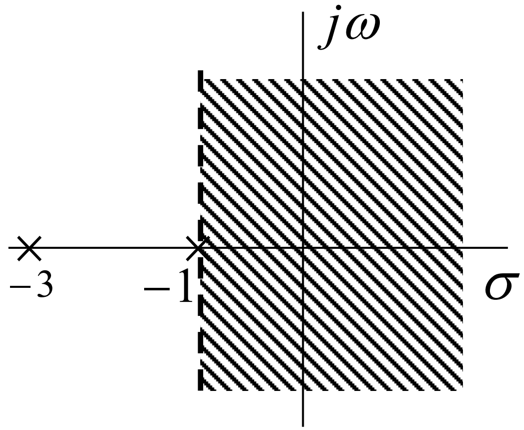
$$\frac{1}{s(2+s)} = \frac{A}{(2+s)} + \frac{B}{s}$$

then $A = \frac{-1}{2}$ and $B = \frac{1}{2}$

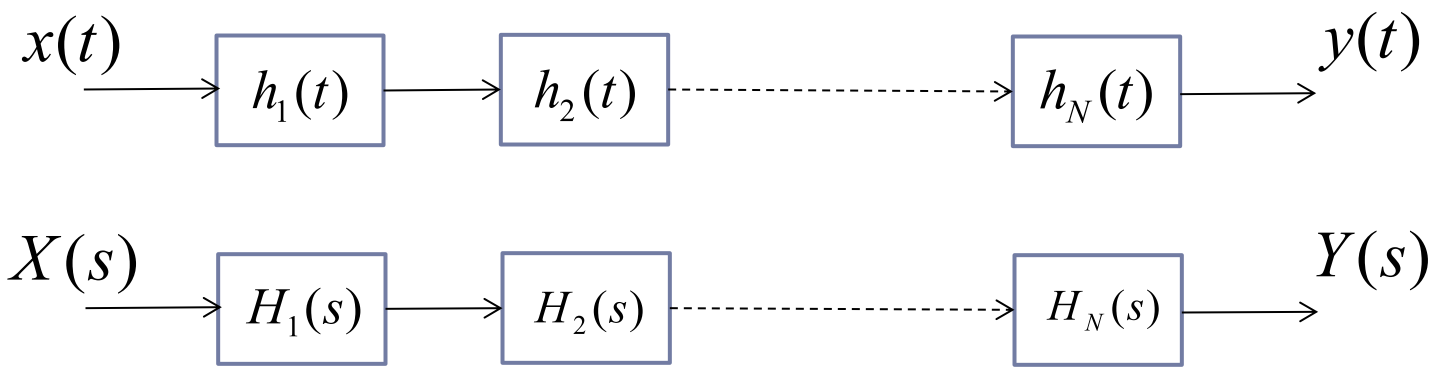
Causality and Stability of LTI Systems:

- The ROC associated with the system function for causal system is a right-half plane.
- An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis.
- A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane (i.e, *all of the poles have negative real parts*).

- Example: the following system is causal and stable



Cascade Connection:



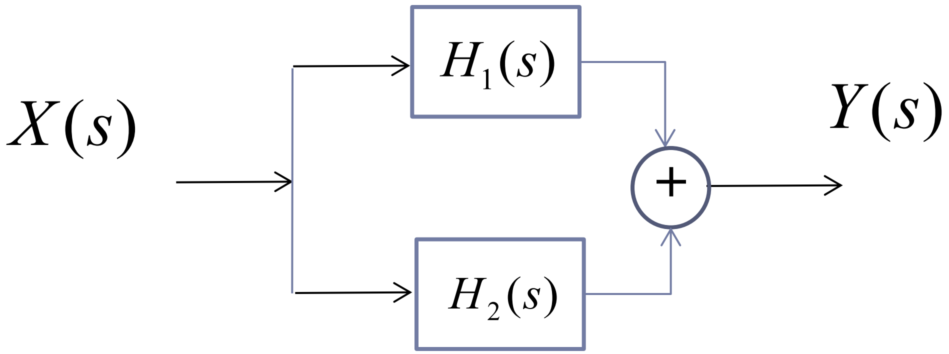
The total impulse response of N-cascade systems is:

$$h(t) = h_1(t) * h_2(t) * \cdots\cdots * h_N(t)$$

and the frequency response of N-cascade systems is:

$$H(s) = H_1(s) \cdot H_2(s) \cdot \cdots\cdots H_N(s)$$

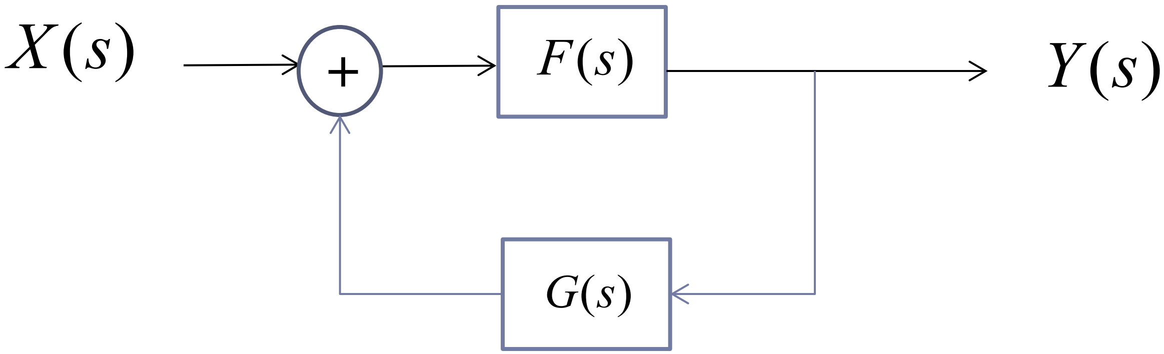
Parallel Connection:



The frequency response of two parallel systems is:

$$H(s) = H_1(s) + H_2(s)$$

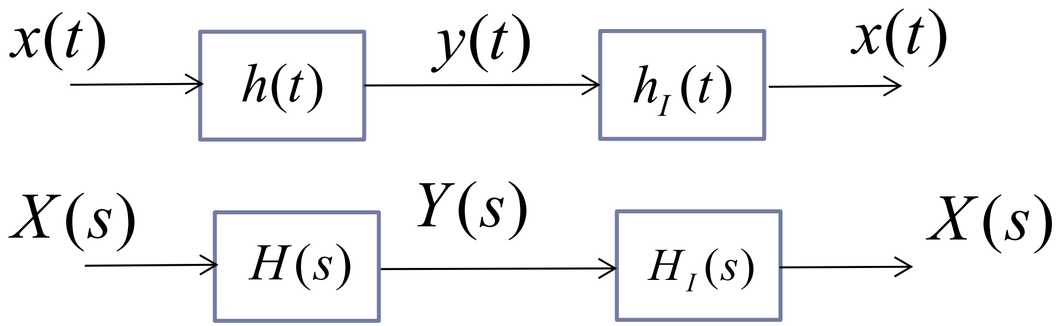
Feedback Connection:



The frequency response (transfer function) of this system is:

$$H(s) = \frac{F(s)}{1 - F(s)G(s)}$$

Inverse Systems:



$$h(t) * h_I(t) = \delta(t)$$

$$H(s) \cdot H_I(s) = 1 \qquad \Rightarrow \qquad H_I(s) = \frac{1}{H(s)}$$

Example:

If $\quad H(s) = \dfrac{s+2}{s+3}$

then $\quad H_I(s) = \dfrac{s+3}{s+2}$

In general: (Differentiation)

$$\frac{d^n x(t)}{dt^n} \stackrel{L}{\longleftrightarrow} s^n X(s)$$

Example: Find the transfer function of the following system

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - 4x(t)$$

Solution:

$$s^2 Y(s) + 3sY(s) + 2Y(s) = sX(s) - 4X(s)$$

$$(s^2 + 3s + 2)Y(s) = (s-4)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-4}{s^2+3s+2}$$

END of Ch.5